Fundamentals (chapter 1)

Basic principle of counting :-

If experiment 1 can result in m possible outcome & for each outcome of experiment 1, there are n possible outcome of experiment 2. Then together there are mn possible outcome of the 2 experiment.

Permutation (arrangement) :-

$$ {}^{n}P_{r} = \frac{n!}{(n-r)!} $$

combination :-

$$ {}^{n}C_{r} = \frac{n!}{(n-r)!\,r!} = \binom{n}{r} $$

The Binomial theorem:

$$ (x+y)^{n} = \sum_{k=0}^{n} \binom{n}{k} x^{k} y^{n-k} $$

Multinomial coefficient:

If $n_1 + n_2 + \cdots + n_r = n$, then

$$ \binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!} $$

represents the number of possible divisions of n distinct objects into r distinct groups of respective size $n_1, n_2, \cdots n_r$.

The Multinomial th^{m} :-

$$ (x_1 + x_2 + \cdots + x_r)^{n} = \sum_{\substack{(n_1, \cdots, n_r): \\ n_1 + \cdots + n_r = n}} \frac{n}{(n_1, n_2, \cdots n_r)!} \binom{n}{n_1, n_2, \cdots n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r} $$

i.e the sum is over all nonnegative integer valued vectors $(n_1, n_2, \cdots, n_r)$ s.t $n_1 + n_2 + \cdots + n_r = n$.

→ The numbers $\binom{n}{n_1, n_2, \cdots, n_r}$ are called as multinomial coefficients.

Axioms of probability:

Sample Space & Event:

When the possible outcomes of an experiment is known in advance, the set of all possible outcomes of an experiment is known as the sample space of the experiment & is denoted by S.

eg/ Flipping of a coin., then

$S = \{H, T\}$

a) Tossing of a dice.

$S = \{1, 2, 3, 4, 5, 6\}$

→ Any subset E of the sample space is known as an event. In other word, an event is a set. consisting of possible outcomes of the experiment of the outcomes of the experiment is contained in E, then we say E has occured.

In eg.1 if $E = \{H\}$, then E is the event that Head appear wther flipping of coin.

Similarly $E = \{T\}$ is also an event that the outcom is Tail.

now $E \cup F = \{H, T\}$

- The event $E \cup F$ is called the union of the event E and event F consist of all outcomes that are either in E or in F or in both E & F.
- Similarly for 2 events E & F, we define EF called the intersection of E & F, consist of all outcomes that are both in E & F.

$E F \equiv E \cap F$

eg/ Let 2 two coints are flipped.

$S = \{(H, H), (H, T), (T, H), (T, T)\}$

E be the event that atleast 1 head occurs

$E = \{(H, H), (H, T), (T, H)\}$

F be the event that atleast 1 tail occurs.

$F = \{(H,T),(T,H),(T,T)\}$

now $EF = \{(H,T),(T,H)\}$ is the event that exactly 1 head & 1 tail occur.

- In eg 1. $E = \{H\}$, $F = \{T\}$

then $EF = \phi$, such an event having no element is called as null event & denoted by ϕ

So if $EF = \phi$, then E & F are said to be mutually exclusive.

Complement of an event

For any event E, we define a new event E^c. referred as the complement of E, to consist of all outcomes in the sample space that are not in E. i.e E^c will occure iff E does not occur.

NOTE

- If E & F are 2 events. If all of the outcomes in E are also in F, then we say that E is contained in F, or E is a subset of F, and write $E \subset F$ (or $F \supset E$ i.e F is superset of E).
- Thus if $E \subset F$ then occurence of E $\Rightarrow$ occurence of F
- If $E \subset F$ & $F \subset E$, then E & F are equal i.e $E = F$

$\Rightarrow$ The above operations of events obeys the following law.

(1) commutative law

$E \cup F = F \cup E$ and $EF = FE$ $(EF = E \cap F)$

(2) Associative law

$(E \cup F) \cup G = E \cup (F \cup G)$ and $(EF)G = E(FG)$

(3) Distributive law

$(E \cup F)G = EG \cup FG$ and $EF \cup G = (E \cup G)(F \cup G)$

Demorgan's Law

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$$

$$\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c$$

[→ for two events E & F, Demorgan's law stated that:

$(E \cup F)^c = E^c F^c$ & $(EF)^c = E^c \cup F^c$]

Pf:- Let x is an outcome of $\left(\bigcup_{i=1}^{n} E_i\right)^c$

$\Rightarrow x$ is not contained in $\bigcup_{i=1}^{n} E_i$

$\Rightarrow x$ is not contained in any of the event E_i, $i = 1, 2, \ldots n$.

$\Rightarrow x$ is contained in $E_i^c \ \forall \ i = 1, 2, \ldots n$

$\Rightarrow x$ is contained in $\bigcap_{i=1}^{n} E_i^c$.

~~Similarly the 2nd part can be proved.~~

$$\Rightarrow \left(\bigcup_{i=1}^{n} E_i\right)^c \subset \bigcap_{i=1}^{n} E_i^c \quad \text{---(1)}$$

now let x is an outcome of $\bigcap_{i=1}^{n} E_i^c$.

$\Rightarrow x$ is contained in $E_i^c \ \forall \ E_i, i = 1, 2, \ldots n$.

$\Rightarrow x$ is not contained in E_i, for any $i = 1, 2, \ldots n$

$\Rightarrow x$ is not contained in $\bigcup_i E_i$.

$\Rightarrow x$ is contained in $\left(\bigcup_i E_i\right)^c$.

$$\Rightarrow \bigcap_{i=1}^{n} E_i^c \subset \left(\bigcup_{i=1}^{n} E_i\right)^c \quad \text{---(2)}$$

So from (1) & (2)

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$$

- Similarly the 2nd law can be proved.

i.e $\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c$

Axioms of Probability:

→ Let us consider an experiment whose sample space is S is repeatedly performed under exactly same cond^n.

Let for each event E of S, n(E) denote the number of times the 1st n repetation of the experiment that the event E occurs.

Then P(E) is the probability of E is defined as

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

→ If the sample space S is finite, the probability of an event E denoted by P(E) is defined as $P(E) = \frac{\text{size of } E}{\text{size of } S}$.

eg: tossing a coin once

$S = \{h, t\}$.

P(E) where E is getting a head $= \frac{1}{2}$

P(F) " F " tail $= \frac{1}{2}$

Axioms:

(1) for any event, $0 \le P(E) \le 1$

(2) $P(S) = 1$

(3) If A, B are mutually exclusive events.

i.e $A \cap B = \phi$

then $P(A \cup B) = P(A) + P(B)$.

It is true for n number of mutually exclusive events also.

Thm:

$P(\phi) = 0$, ϕ is empty set.

Pf// Let A be any event s.t subset of S.

$\because A \cap \phi = \phi$,

A & ϕ are mutually exclusive event,

$\therefore P(A) = P(A \cup \phi) = P(A) + P(\phi) \Rightarrow P(\phi) = 0$

Ex-2 : Proposition 4.1

$\boxed{P(A^c) = 1 - P(A)}$. where A^c is complement of A in S.

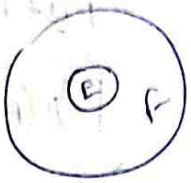
Pf/ $\because A$ & A^c are mutually exclusive.

$P(A) + P(A^c) = P(A \cup A^c) = P(S) = 1$

$\Rightarrow P(A^c) = 1 - P(A)$.

Proposition 4.2

If $E \subset F$, then $P(E) \leq P(F)$



Pf/ Since $E \subset F$,

so $F = E \cup (E^c \cap F) = E \cup (E^c F)$

Hence, since E & $E^c F$ are mutually exclusive.

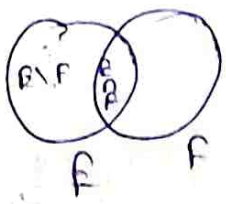
$P(F) = P(E \cup E^c F) = P(E) + P(E^c F)$

Since $P(E^c F) \geq 0$

so $P(E) \leq P(F)$. □

Ex :-

$P(E - F) = P(E) - P(E \cap F)$.



Pf/ $E = (E - F) \cup (E \cap F)$

Sine $E - F$ & $E \cap F$ are two disjoint sets. By axiom 3.

$P(E) = P(E - F) + P(E \cap F)$

$\Rightarrow P(E - F) = P(E) - P(E \cap F)$. □

Proposition 4.3

$P(E \cup F) = P(E) + P(F) - P(EF)$

Pf/ $E \cup F$ can be written as the union of 2 disjoint events E & $E^c F$.

so $P(E \cup F) = P(E \cup E^c F)$